Def: For a set $\Omega \subseteq \mathbb{R}^n$, define $L_\Omega(r) = \#(r\Omega \cap \mathbb{Z}^n)$

Thm: Let $\Omega \subseteq \mathbb{R}^n$ be bounded and jordan measurable. Then $\lim_{r\to\infty} \frac{L_\Omega(r)}{r^n} = Vol(\Omega)$

Guess: For a general lattice, Λ, the limit is probably $\frac{Vol(\Omega)}{Covol(\Lambda)}$

Pf: Note also that $L_\Omega(r) = \#\left(\Omega \cap \frac{1}{r}\mathbb{Z}^n\right)$

Ω bounded, so it is contained in a cube $C = [-M, M]^n$

Subdivide the cube into $(2Mr)^n$ cubes of side length $\frac{1}{r}$ and note that adding up the volumes of each cube whose corner $(\frac{x_i}{r})_{i=1}^n$ is in Ω gives a Riemann sum for $vol(\Omega)$

But the volume of each cube is $\frac{1}{r^n}$ and there are $L_\Omega(r)$ such cubes, hence the volume is $\frac{L_\Omega(r)}{r^n}$

Taking limits as $r \to \infty$ gives the result.

For general Λ we have a change of basis matrix

$$T : \mathbb{Z}^N \to \Lambda, \quad |\det(T)| = \mathrm{covol}(\Lambda)$$

$$X \in \Omega \cap \tfrac{1}{r}\Lambda \iff T^{-1}X \in T^{-1}\Omega \cap \tfrac{1}{r}\mathbb{Z}^N$$

$$\frac{L_{\Omega,\Lambda}(r)}{r^N} = \frac{L_{T^{-1}\Omega}(r)}{r^N} \to \mathrm{vol}(T^{-1}\Omega) = \frac{\mathrm{vol}(\Omega)}{|\det(T)|}$$

$$= \frac{\mathrm{vol}(\Omega)}{\mathrm{covol}(\Lambda)}$$

Thm: Let $\Omega \subseteq \mathbb{R}^N$ be convex, symmetric with $\frac{\mathrm{vol}(\Omega)}{\mathrm{covol}(\Lambda)} > 2^N$. Then $\Omega \cap \Lambda \supsetneq \{0\}$

Pf: It suffices to show that the translates $\frac{1}{2}\Omega + \lambda$ are not all disjoint

↳ once we have this $\exists \lambda_1 \neq \lambda_2 \in \Lambda : \frac{1}{2}\Omega + \lambda_1 \cap \frac{1}{2}\Omega + \lambda_2 \neq \emptyset$

Let $y = \frac{1}{2}x_1 + \lambda_1 = \frac{1}{2}x_2 + \lambda_2$

Then $z = \frac{1}{2}x_1 - \frac{1}{2}x_2 = \lambda_2 - \lambda_1 \in \Lambda \setminus \{0\}$

$$x_2 \in \Omega \to -x_2 \in \Omega \\ x_1 \in \Omega \Big\} \to \tfrac{1}{2}x + \tfrac{1}{2}(-x_2) \in \Omega$$

Hence $z \in \Omega \cap \Lambda \setminus \{0\}$

Suppose by contradiction all such translates are disjoint

Then so too are $\frac{1}{2}\Omega + \lambda \cap F$ where F is fundamental parallelepiped for Λ

Hence, $\sum_{\lambda} \mathrm{vol}\left(\frac{1}{2}\Omega + \lambda \cap F\right) \leq \mathrm{vol}(F)$

The $F-\lambda$ cover $\mathbb{R}^n$, so we get

$$\frac{\mathrm{vol}(\Omega)}{2^n} = \mathrm{vol}\left(\frac{1}{2}\Omega\right) = \sum_{\lambda} \mathrm{vol}\left(\frac{1}{2}\Omega \cap F - \lambda\right)$$

$$= \sum_{\lambda} \mathrm{vol}\left(\frac{1}{2}\Omega + \lambda \cap F\right)$$

$$\leq \mathrm{vol}(F) = \mathrm{covol}(\Lambda)$$

Hence $\frac{\mathrm{vol}(\Omega)}{\mathrm{covol}(\Lambda)} \leq 2^n$ ↯

Def: For a lattice $\Lambda \subseteq \mathbb{R}^n$ and a convex $\Omega \subseteq \mathbb{R}^n$, define for $1 \leq i \leq n$ the successive minima of Λ w.r.t. Ω as

$\lambda_i(\Omega, \Lambda) := \inf\{r > 0 : r\Omega \cap \Lambda$ contains i linearly independent pts$\}$

Thm (Minkowski's Second Thm): Let Ω be convex and bounded and Λ a lattice. Then

$$\frac{2^N}{N!}\prod_{i=1}^{N}\frac{1}{\lambda_i(\Omega,\Lambda)} \le \frac{\mathrm{vol}(\Omega)}{\mathrm{covol}(\Lambda)} \le 2^N \prod_{i=1}^{N}\frac{1}{\lambda_i(\Omega,\Lambda)}$$

Equivalently, take reciprocals and mult. by 2^N for

$$\prod_{i=1}^{N}\lambda_i(\Omega,\Lambda) \le \frac{2^N\,\mathrm{covol}(\Lambda)}{\mathrm{vol}(\Omega)} \le N!\prod_{i=1}^{N}\lambda_i(\Omega,\Lambda)$$

Where Minkowski's Thms are Used

(1) · Finiteness of the class number

· Linear Forms Thm

· Approximation

↳ $\forall \alpha\in\mathbb{R}, Q>1$, $\exists\, p,q\in\mathbb{Z}$, $|q|<Q$:

$$|\alpha q - p| < \frac{1}{Q}$$

· Four Square Thm.

(2) Roy and Thunder